International Journal of Theoretical Physics, Vol. 29, No. 4, 1990

Dirac and Maxwell Equations in the Clifford and Spin-Clifford Bundles

W. A. Rodrigues, Jr.[1] **and E. Capelas de Oliveira**[1]

Received June 30, 1989

We show how to write the Dirac and the generalized Maxwell equations (including monopoles) in the Clifford and spin-Clifford bundles (of differential forms) over space-time (either of signature $p=1$, $q=3$ or $p=3$, $q=1$). In our approach Dirac and Maxwell fields are represented by objects of the same mathematical nature and the Dirac and Maxwell equations can then be directly compared. We show also that all presentations of the Maxwell equations in (matrix) Dirac-like "spinor" form appearing in the literature can be obtained by choosing particular global idempotents in the bundles referred to above. We investigate also the transformation laws under the action of the Lorentz group of Dirac and Maxwell fields (defined as algebraic spinor sections of the Clifford or spin-Clifford bundles), clearing up several misunderstandings and misconceptions found in the literature. Among the many new results, we exhibit a factorization of the Maxwell field into two-component spinor fields (Weyl spinors), which is important.

1. INTRODUCTION

There are several presentations of the Maxwell equations in (matrix) Dirac-like "spinor" form in the literature. The motivations for these presentations are: (i) to give a quantum mechanical (first quantization) interpretation to the Maxwell field; and (ii) the belief[2] that spinors are more fundamental objects than tensors. However, these presentations, although very ingenious, are of an *ad hoc* nature and do not leave clear, among other issues, which is the transformation law of the spinor representing the Maxwell field under the action of the Lorentz group.

[1]Departamento de Matemática Aplicada, Instituto de Matemática, Estatística e Ciência da Computação, Universidade Estadual de Campinas-UNICAMP, 13081 (Campinas) São Paulo, Brazil.

[2]Wrong, according to the point of view of this paper. See also Figueiredo *et al.* (1990) and Rodrigues and Figueiredo (1990).

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The importance of representing the electromagnetic field and the matter field as objects of the same mathematical nature is obviously an important step for any tentative construction of a unified theory.

One of the purposes of the present paper is to clarify that despite the fact that both fields can be represented as objects of the same mathematical nature, they satisfy distinct field equations even if these can look formally identical in some particular representations. This point, in particular, is at variance with the claim of Sallhöfer (1986), who proposes that both fields should be identified.

To attain our objective, we show in Section 2 how to write the usual Dirac equation written for a covariant Dirac spinor field as an equation for an algebraic Dirac spinor field[3] in the Clifford and spin-Clifford bundles of differential forms over space-time. This implies in the use of $\mathbb{R}_{1,3}$ the space-time and in $\mathbb{R}_{3,1}$ the Majorana algebras (Figueiredo *et al.*, 1990). We do not use here $\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, the Dirac algebra. The definitions and the properties of these bundles necessary for the present paper are reviewed in Section 4 (see also Rodrigues and Figueiredo, 1989).

In what follows $\mathscr{L}$ denotes a Minkowski space of signature $p = 1$, $q = 3$, i.e., a triple $(\mathscr{M}, g, \nabla)$, where $\mathscr{M}$ is a 4-dimensional manifold, connected, noncompact, time-oriented, and space-time-oriented, g is the Lorentz metric, and ∇ is the Levi-Civita connection of g in $\mathscr{M}$ (Rodrigues and Faria-Rosa, 1989). $\bar{\mathscr{L}}$ denotes a Minkowski space of signature $p = 3$, $q = 1$. (The case of the formulation of the Dirac and Maxwell equations within the Clifford and spin-Clifford bundle formalism in a general Lorentzian manifold is presented in a separate paper.)

We denote the respective Clifford and spin-Clifford bundles corresponding to $\mathscr{L}$ ($\bar{\mathscr{L}}$) by $\mathscr{C}\ell(\mathscr{L})$ [$\mathscr{C}\ell(\bar{\mathscr{L}})$] and $S\mathscr{C}\ell$ ($\mathscr{L}$) [$S\mathscr{C}\ell(\bar{\mathscr{L}})$].

Among our results, we show that the Dirac field originally interpreted as an algebraic spinor section of $\mathscr{C}\ell(\mathscr{L})$ [or $S\mathscr{C}\ell(\dot{\mathscr{L}})$] obeys a differential equation where the global idempotent field (defining the ideal section of the bundle to where the Dirac field leaves) factors out.

We then obtain an equation for a Clifford field that is a sum of a 0-form plus 2- plus 4-form fields. It is quite interesting that this object is analogous to the generalized electromagnetic field describing charges and (nontopological) monopoles as obtained in (Faria-Rosa and Rodrigues, 1989; Faria-Rosa *et al.*, 1986; Rodrigues *et al.*, 1988, 1989*a*,*b*).

In Section 3 we show how to write the Maxwell equations [even for the case containing (nontopological) magnetic monopoles] in the Clifford and spin-Clifford bundles, where the electromagnetic field is represented as an algebraic Dirac spinor field. Then we show that all presentations of

[3]The precise definitions of these concepts can be found in Figueiredo *et al.* (1990) and Rodrigues and Figueiredo (1989, 1990).

the Maxwell equations in (matrix) Dirac-like form in the literature can be obtained by particular choices of the global idempotent in the Clifford or spin-Clifford bundles.

A particular choice of the global idempotent [that is simultaneously a local idempotent of both $\mathbb{R}_{1,3}$ and $\mathbb{R}_{3,0}^{+}$—the Pauli algebra (Rodrigues *et al.*, 1989*a*)] permits us to write the Maxwell equations as two equations satisfied by Weyl spinor fields. We can show that in this formulation new invariants of the Maxwell field show up. The importance of this result will be discussed in a separate paper.

In Section 4 we discuss the transformation laws of the Dirac algebraic spinor fields and the Maxwell algebraic spinor field both in $\mathscr{C}\ell(\mathscr{L})$ and $S\mathscr{C}\ell(\mathscr{L})$, clearing up several misconceptions in the literature.

Finally in Section 5 we present our conclusions.

2. THE DIRAC EQUATION IN $\mathscr{C}\ell(\mathscr{L})$ AND $S\mathscr{C}\ell(\mathscr{L})$

In the usual presentation of the Dirac equation, the (standard) covariant Dirac spinor field (Landau and Lifschitz, 1971; Bleecker, 1971) is assumed to be a section $\Psi(x)=(x, \check{\psi}(x))$ of the covariant spinor bundle $\bar{S}(\mathscr{L})= P_{Spin_+(1,3)}(\mathscr{L})\times_{\bar{\rho}} V$, where $P_{Spin_+(1,3)}(\mathscr{L})$ is the covariant *spinor structure bundle* (Bleecker, 1971) and $\bar{\rho}$: $Spin_+(1,3)\to GL(V)$ is the $D^{(1/2,0)}\oplus D^{(0,1/2)}$ representation of $SL(2,\mathbb{C})\simeq Spin_+(1,3)$, the universal covering group of $SO_+(1,3)$, the homogeneous Lorentz group, and $V=\mathbb{C}^4$, a 4-dimensional complex vector space. The standard Dirac covariant spinors are elements of $\mathbb{C}^4$ constructed in the following way:

$$\mathbb{C}^4\ni\check{\psi}=\begin{pmatrix}\phi\\ \lambda\end{pmatrix} \tag{1}$$

$$\phi=\frac{1}{\sqrt{2}}(\xi+\overset{\Delta}{\dot{\eta}});\qquad \lambda=\frac{1}{\sqrt{2}}(\xi-\overset{\Delta}{\dot{\eta}})$$

where $\xi\in\mathbb{C}^2$ (a 2-dimensional complex vector space) are the so-called undotted two-component Weyl spinors and $\overset{\Delta}{\dot{\eta}}\in\overset{\Delta}{\dot{\mathbb{C}}}{}^2$, where $\dot{\mathbb{C}}^2$ is also a 2-dimensional complex vector space) and is the algebraic dual of dotted two-component Weyl spinor relative to the spinorial metric (Figueiredo *et al.*, 1990).

The Dirac matrices $\boldsymbol{\gamma}_\mu\in\mathbb{C}(4)$, $\mu=0,1,2,3$, act as linear operators in $\mathbb{C}^4$ and have the representation

$$\check{\boldsymbol{\gamma}}_0=\begin{bmatrix}\mathbb{1}_1 & 0\\ 0 & -\mathbb{1}_2\end{bmatrix};\qquad \check{\boldsymbol{\gamma}}_k=\begin{bmatrix}0 & -\sigma_k\\ \sigma_k & 0\end{bmatrix};\qquad \check{\boldsymbol{\gamma}}_5=\check{\boldsymbol{\gamma}}_0\check{\boldsymbol{\gamma}}_1\check{\boldsymbol{\gamma}}_2\check{\boldsymbol{\gamma}}_3=\begin{bmatrix}0 & i\mathbb{1}_2\\ i\mathbb{1}_2 & 0\end{bmatrix} \tag{2}$$

$$\check{\boldsymbol{\gamma}}_\mu\check{\boldsymbol{\gamma}}_\nu+\check{\boldsymbol{\gamma}}_\nu\check{\boldsymbol{\gamma}}_\mu=2\eta_{\mu\nu};\qquad \eta_{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1) \tag{3}$$

and σ_i, $i=1,2,3$, are the Pauli matrices and $\mathbb{1}_2$ is the 2-dimensional unit matrix.

The coupling of the standard Dirac covariant spinor field and the electromagnetic field represented by the vector potential A, a section of the cotangent bundle $T^*\mathscr{L}$, is then given by

$$i\check{\gamma}^\mu(\partial_\mu - qA_\mu(x))\check{\psi}(x) = m\check{\psi}(x) \tag{4}$$

where the pair (q, m) represents, respectively, the charge and the mass of the Dirac field.

Let $\mathbb{R}^{1,3}$ be a Minkowski *vector* space (Rodrigues and Faria-Rosa, 1989) (not to be cofounded) with $\mathscr{L}$, E_μ, $\mu=0,1,2,3$, the canonical basis and g the Lorentz metric in $\mathbb{R}^{1,3}$, such that $g(E_\mu, E_\nu)=\eta_{\mu\nu}$. The *real* Clifford algebra $\mathbb{R}_{1,3}$ (nowhere called the space-time algebra) (Figueiredo *et al.*, 1990; Rodrigues *et al.*, 1989*a*) is generated by the E_μ. Taking $\mathbb{R}^{1,3}$ canonically imbedded in $\mathbb{R}_{1,3}$, we have

$$E_\mu E_\nu + E_\nu E_\mu = 2g(E_\mu, E_\nu) = 2\eta_{\mu\nu} \tag{5}$$

analogous to equation (3). Let $\mathbb{R}^+_{1,3} \simeq \mathbb{R}_{3,0}$ (the Pauli algebra) be the even subalgebra of $\mathbb{R}_{1,3}$. Now, $\check{E}=\frac{1}{2}(1+E_0)$ is a primitive idempotent of $\mathbb{R}_{1,3}$.[4] As for any $x\in\mathbb{R}_{1,3}$ there exists $y\in\mathbb{R}^+_{1,3}$ such that $x\check{E}=y\check{E}$, it follows that $\check{I}=\mathbb{R}^+_{1,3}\check{E}$ is a minimal left ideal of $\mathbb{R}_{1,3}$.

Every $\check{\psi}\in\check{I}$ can be written in the form

$$\check{\psi} = \check{E}\psi_1 + E_3E_1\check{E}\psi_2 + E_3\check{E}\psi_3 + E_1\check{E}\psi_4 \tag{6}$$

where $\psi_1 \in \check{E}\mathbb{R}_{1,3}\check{E} \simeq \mathbb{C}$ with basis $\{1, E_2E_1\}\check{E}$, and $\check{\alpha}=\{\check{E}, E_3E_1\check{E}, E_3\check{E}, E_1\check{E}\}$ is a complex spinorial basis for $\check{I}$.

The $\check{\psi}$ (as the notation anticipates) are representatives of the standard Dirac covariant spinors introduced through (1) and will be called standard algebraic Dirac spinors (or algebraic spinors, for short, when no confusion arises). This can be seen once we consider the isomorphism

$$\begin{aligned} \check{\gamma}:\quad & \mathbb{R}_{1,3} \to \mathscr{L}_{\mathbb{C}}(\check{I}) \\ & p \to \check{\gamma}(p):\quad \check{I} \to I \\ & \qquad\qquad\qquad \check{\psi} \to p\check{\psi} \end{aligned} \tag{7}$$

In (7), $\mathscr{L}_{\mathbb{C}}(\check{I})$ is the space of linear transformations of $\check{I}$ (over the complex field). The isomorphism $\check{\gamma}$ gives, through a technique introduced

[4]For details of the notation see Figueiredo *et al.* (1990), Rodrigues and Figueiredo (1990), and Rodrigues *et al.* (1989*a*).

in Figueiredo *et al.* (1990), for $\check{\gamma}(E_\mu)=\check{\boldsymbol{\gamma}}_\mu$, $\mu=0,1,2,3$, and $\check{\gamma}(E_5)=\check{\boldsymbol{\gamma}}_5$ exactly the set of Dirac matrices exhibited in (2). We also have

$$\check{\gamma}(\check{E})=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} \tag{8}$$

Putting $\check{\gamma}(\check{\alpha})=\{|1\rangle,|2\rangle,|3\rangle,|4\rangle;\ |i\rangle\in\mathbb{C}(4)\gamma(\check{E}),\ i=1,2,3,4\}$, we obtain the validity of the following identities:

$$\begin{gathered}|1\rangle=\check{\boldsymbol{\gamma}}_0|1\rangle;\qquad i|1\rangle=\check{\boldsymbol{\gamma}}_2\check{\boldsymbol{\gamma}}_1|1\rangle;\qquad |2\rangle=-\check{\boldsymbol{\gamma}}_5\check{\boldsymbol{\gamma}}_2|1\rangle\\ |3\rangle=\check{\boldsymbol{\gamma}}_3|1\rangle;\qquad |4\rangle=\check{\boldsymbol{\gamma}}_1|1\rangle\end{gathered} \tag{9}$$

Given this local structure, we can now map the Dirac equation written for the standard Dirac covariant spinor, i.e., equation (4) in the Clifford bundle of differential forms $\mathscr{C}\ell(\mathscr{L})$, once we take into account one more result of purely mathematical character. As $\mathscr{C}\ell(\mathscr{L})$ is the quotient of the tensor bundle by an appropriate ideal, then the Levi-Civita connection passes to the quotient and can be used in $\mathscr{C}\ell(\mathscr{L})$ (Graf, 1978; Blaine Lawson and Michelsohn, 1983). Let now $\{e_\mu,\ \mu=0,1,2,3\}$ be a global tetrad field, i.e., a section of a frame bundle. Let $\{\gamma^\mu,\ \mu=0,1,2,3\}$ be a section of the coframe bundle[5] dual to $\{e_\mu\}$, i.e., $\gamma^\mu(e_\nu)=\delta^\mu_\nu$. In the Clifford bundle $\mathscr{C}\ell(\mathscr{L})$ of differential forms we give the structure of a Clifford algebra to each $T^*\mathscr{L}\simeq\mathbb{R}^{1,3}$. In $\mathscr{C}\ell(\mathscr{L})$ we have the global idempotent field $\check{e}=\frac{1}{2}(1+\gamma^0)$ of global minimum rank 8 (Rodrigues and Figueiredo, 1989) and the algebraic spinor field $\Psi=(x,\check{\psi}(x))$ corresponding to the standard covariant Dirac spinor field is defined as a section of $\mathscr{C}\ell(\mathscr{L})$ such that $\check{\psi}\check{e}=\check{\psi}$. Putting $\partial=\gamma^\mu\partial_\mu$ (the Dirac operator), we can show that (Rodrigues *et al.*, 1989*a*)

$$\partial=d-\delta \tag{10}$$

where d is the differential and δ is the Hodge codifferential. Using this last result plus equations (6) and (9), the Dirac equation in $\mathscr{C}\ell(\mathscr{L})$ results,

$$[\partial\mathscr{C}(x)\check{\gamma}^2\check{\gamma}^1-qA(x)\mathscr{C}(x)]\check{\gamma}^0\check{e}=m\mathscr{C}(x)\check{e} \tag{11}$$

where $\mathscr{C}(x)\in\sec\Lambda^0T^*\mathscr{L}+\Lambda^2T^*\mathscr{L}+\Lambda^4T^*\mathscr{L}$, i.e., locally $\mathscr{C}(x)$ is an element of $\mathbb{R}^+_{1,3}$ and $\check{\psi}(x)=\mathscr{C}(x)\check{e}$. Also, $A(x)=\gamma^\mu A_\mu(x)$.

We observe that in equation (11), $i=\sqrt{-1}$ has been eliminated! This point is very important. It shows that—contrary to the approach by Graf (1978), who uses the Kähler equation in $\mathscr{C}\ell(\mathscr{L})$—here there is no need for

[5]Note that the basis vectors of the coframe bundle are represented by Greek letters γ^μ without boldface. Boldface characters are reserved for Dirac gamma matrices.

the complexification of $\mathscr{C}\ell(\mathscr{L})$ by $U(1)$-gauging.[6] Observe also that in (11) the global idempotent field can be "factored" out, thus resulting in an equation for $\mathscr{C}(x)$. Hestenes (1967, 1971*a,b*, 1975, 1986) calls this object an "operator spinor," but it is clear from the detailed analysis in Figueiredo *et al.* (1990) and Rodrigues and Figueiredo (1990) that this object is not an algebraic spinor field. $\mathscr{C}(x)$ is incidentally an object of the same mathematical nature as the generalized electromagnetic field generated by charges and (nontopological) monopoles (Rodrigues *et al.*, 1989; Faria-Rosa and Rodrigues, 1989). It is also important to observe that the equation satisfied by $\mathscr{C}(x)$ is different from the Kähler equation (Graf, 1978), which does not contain the term $\gamma^2\gamma^1$.

The elimination of $\sqrt{-1}$ gives rise to a geometrical and realistic interpretation of the Dirac equation, as discussed in Hestenes (1967, 1971*a,b*, 1975, 1985) and Guèret (1989). It is important here to emphasize that the derivation of (11) in Hestenes (1967, 1971*a,b*, 1975) is *ad hoc.* The reason is that $\mathbb{R}_{1,3}\simeq\mathbb{H}(2)$, where $\mathbb{H}$ is the quaternion field, and so the elements of minimal left ideals $(\mathbb{R}_{1,3}\check{E})$ cannot be directly identified with the standard Dirac covariant spinors. This can be done only using the complex spinorial basis as above.

Now, $E=\frac{1}{2}(1+E_3E_0)$ is also a primitive idempotent of $\mathbb{R}_{1,3}$ (and also a primitive idempotent of $\mathbb{R}^+_{1,3}\simeq\mathbb{R}_{3,0}$). The minimal ideal $I=\mathbb{R}_{1,3}E$ is such that its elements represent Dirac covariant spinors where the representation space V in the definition of the covariant spinor bundle $\bar{S}(\mathscr{L})$ has the structure $V=\mathbb{C}^2\oplus\overset{\Delta}{\dot{\mathbb{C}}}$ (Figueiredo *et al.*, 1990; Bleecker, 1971). The complex spinorial basis in this case is $\alpha=(E_0E, E_1E, E, E_0E_1E)$, and we have

$$I\ni\phi=E_0E\phi_1+E_1E\phi_2+E\phi_3+E_0E_1E\phi_4 \tag{12}$$

with $\phi_i\in E\mathbb{R}_{1,3}E\simeq\mathbb{C}$ with basis $\{1, E_5\}E$. Considering the injection

$$\begin{aligned}\gamma:\quad &\mathbb{R}_{1,3}\to\mathscr{L}_{\mathbb{C}}(I)\\ &p\to\gamma(p):\quad I\to I\\ &\qquad\qquad\qquad \phi\to p\phi\end{aligned} \tag{13}$$

we get the following representation for E and E_μ, $\mu=0,1,2,3$, in the α-basis (which we will need in Section 3):

$$\begin{aligned}\gamma(E)&=\begin{bmatrix}\mathbb{1}_2 & 0\\ 0 & 0\end{bmatrix}, &\gamma(E_0)&=\boldsymbol{\gamma}_0=\begin{pmatrix}0 & \mathbb{1}_2\\ \mathbb{1}_2 & 0\end{pmatrix}\\ \gamma(E_i)&=\boldsymbol{\gamma}_i=\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}, &\gamma(E_5)&=\boldsymbol{\gamma}_5=\begin{pmatrix}-i\mathbb{1}_2 & 0\\ 0 & i\mathbb{1}_2\end{pmatrix}\end{aligned} \tag{14}$$

[6]In the complexification of $\mathscr{C}\ell(\mathscr{L})$, the typical fiber becomes $\mathbb{R}_{4,1}\simeq\mathbb{C}(4)$, the Dirac algebra. The even subalgebra of $\mathbb{R}_{4,1}$ is denoted $\mathbb{R}^+_{4,1}\simeq\mathbb{R}_{1,3}$.

Consider now $\phi \in \sec \mathscr{C}\ell(\mathscr{L})$ such that $\phi e = \phi$ with $e = \frac{1}{2}(1 + \check{\gamma}^3 \check{\gamma}^0)$ a global primitive idempotent field of minimal rank 8. Multiplying both members of (11) by e, we get the Dirac equation in the form

$$\partial\phi\, \check{\gamma}^5 - qA\phi = m\phi; \qquad \phi = \phi e \in \sec \mathscr{C}\ell(\mathscr{L}) \tag{15}$$

which appears originally in Hestenes (1967, 1971*a*,*b*, 1975).

Dirac Equation in $\mathscr{C}\ell(\bar{\mathscr{L}})$ and in $S\mathscr{C}\ell(\mathscr{L})$

As $\bar{\mathscr{L}}$ has the signature $p = 3$, $q = 1$, the associated Clifford algebra is $\mathbb{R}_{3,1} \simeq \mathbb{R}(4)$, the Majorana algebra, generated by $\bar{E}_\mu$, $\mu = 0, 1, 2, 3$, satisfying $\bar{E}_\mu \bar{E}_\nu + \bar{E}_\nu \bar{E}_\mu = -2\eta_{\mu\nu}$. Now, $\bar{E} = \frac{1}{2}(1 + \bar{E}_3 \bar{E}_0)$ is idempotent, but not a minimal one. We can show that Dirac covariant spinors are represented by the elements of the ideal $\bar{I} = \mathbb{R}_{3,1} E$, and have the structure

$$\bar{I} \ni \bar{\psi} = \varphi + \bar{E}_0 \overset{\Delta}{\dot{\chi}} \tag{16}$$

where φ is a representative of Weyl's undotted spinors and $\overset{\Delta}{\dot{\chi}}$ is a Weyl dotted spinor in $\mathbb{R}_{3,1}$ (Figueiredo *et al.*, 1990). The representation of Dirac's equation in $\mathscr{C}\ell(\mathscr{L})$ can now be obtained in an analogous way as done above, but we are not going to develop it here, since we do not need the result for what follows.

What is important is to conclude this section with the remark that the form of the Dirac equation in $S\mathscr{C}\ell(\mathscr{L})$ is identical to equation (11) [or equation (15)] with $\check{\psi}\check{e}$ or $\check{\phi}$ as appropriate sections of $S\mathscr{C}\ell(\mathscr{L})$. This can be seen at once if we keep in mind the close relation between the two bundles (see Appendix) when $\mathscr{L}$ is a manifold that admits a spinor structure, as in our case. However, the transformation laws for $\psi = \psi\check{e} \in \sec \mathscr{C}\ell(\mathscr{L})$ and $\psi = \psi\check{e} \in \sec S\mathscr{C}\ell(\mathscr{L})$ are quite different, as we show in Section 4.

3. MAXWELL EQUATIONS IN $\mathscr{C}\ell(\mathscr{L})$

Let J_e, F, and J_m, respectively, be a 1-form, a 2-form, and a 1-form field defined on $\mathscr{L}$ [i.e., sections of the Hodge bundle canonically imbedded in $\mathscr{C}\ell(\mathscr{L})$ (Rodrigues *et al.*, 1989). J_e is the electric charge current, J_m is the (nontopological) magnetic charge current, and F is the generalized electromagnetic field generated by charges and monopoles. With these definitions we easily show that the generalized Maxwell equations can be written as (Rodrigues *et al.*, 1989)

$$dF = -\star J_m; \qquad \delta F = -J_e \tag{17}$$

where ★ is the Hodge star operator. Now using equation (10) and taking into account that $\forall f_p \in \sec \Lambda^p(T^*\mathscr{L}) \subset \sec \mathscr{C}\ell(\mathscr{L})$, one has (Rodrigues *et al.*, 1989).

$$\star f_p = (-)^t \gamma^5 f_p$$

where $t = 1$ for $p = 1, 2, 3$ and $t = 2$ for $p = 0, 4$, we can write the difference of equations in (17) as

$$\partial F = J_e + \gamma^5 J_m = J \tag{18}$$

Now, if $\mathbf{e}$ is a generic global idempotent field in $\mathscr{C}\ell(\mathscr{L})$, we obtain after multiplication of both members of (18) by e,

$$\partial \psi = \chi; \qquad \psi = F\mathbf{e} \qquad \chi = J\mathbf{e} \tag{19}$$

with $\psi e = \psi$, $\chi e = \chi \in \sec \mathscr{C}\ell(\mathscr{L})$. Equations (19) represent a "Maxwell equation" written in algebraic spinor form.

Taking, e.g., $\mathbf{e} = e$, we can now directly compare equations (15) (for the free-field case) and (19) and we see immediately that they are quite different.

Maxwell Equations in (Matrix) Dirac Covariant Spinor Form

In order to obtain the Maxwell equations (with $J_m = 0$) in the (matrix) covariant "spinor" form as found in the literature, all we have to do is to choose appropriate global idempotent fields in $\mathscr{C}\ell(\mathscr{L})$.

(*i*) For $\mathbf{e} = e = \frac{1}{2}(1 + \gamma^3\gamma^0)$, we get (the sign "$\simeq$" denotes isomorphic)

$$\psi = Fe \simeq \begin{pmatrix} -E_3 + iB_3 & 0 & 0 & 0 \\ -E_1 - iE_2 + iB_1 - B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_1 - iE_2 + iB_1 + B_2 \\ 0 & 0 & 0 & -E_3 - iB_3 \end{pmatrix} \tag{20}$$

$$\chi = Je \simeq \begin{pmatrix} 0 & 0 & 0 & -J_1 + iJ_2 \\ 0 & 0 & 0 & J_0 + J_3 \\ J_0 + J_3 & 0 & 0 & 0 \\ J_1 + iJ_2 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

Putting now

$$\begin{aligned} \varphi_1 &= \begin{pmatrix} -E_3 + iB_3 \\ -E_1 - iB_2 + iB_1 - B_2 \end{pmatrix}; \qquad \xi_1 = \begin{pmatrix} J_0 + J_3 \\ J_1 + iJ_2 \end{pmatrix} \\ \varphi_2 &= \begin{pmatrix} E_1 - iE_2 + iB_1 + B_2 \\ -E_3 - iB_3 \end{pmatrix}; \qquad \xi_2 = \begin{pmatrix} -J_1 + iJ_2 \\ J_0 + J_3 \end{pmatrix} \end{aligned} \tag{22}$$

the equation $\partial Fe = Je$ decouples [under the mapping γ defined in (7)] in this matrix representation into two equations for the Weyl spinors (Rodriguez *et al.*, 1989) of (22), namely

$$\sigma^{\mu}\partial_{\mu}\varphi_{\alpha} = \xi_{\alpha} \qquad (\alpha = 1, 2) \tag{23}$$

Equation (23) was first obtained by Sachs (1982) in an *ad hoc* way starting from a noncovariant equation. Here all the formalism is intrinsically covariant and the presentation of (23) is obtained in a legitimate way. Equation (23) is important for exhibiting new invariants of the Maxwell fields which have very important physical consequences. We treat this point in another paper.

(*ii*) For $\mathbf{e} = \check{e} = \frac{1}{2}(1+\gamma^0)$, we get

$$\psi = F\check{e} \simeq \begin{pmatrix} iB_3 & iB_1 + B_2 & 0 & 0 \\ iB_1 - iB_2 & -iB_3 & 0 & 0 \\ E_3 & E_1 - iE_2 & 0 & 0 \\ E_1 + iE_2 & -E_3 & 0 & 0 \end{pmatrix} \tag{24}$$

$$\xi = J\check{e} \simeq \begin{pmatrix} J_0 & 0 & 0 & 0 \\ 0 & J_0 & 0 & 0 \\ J_3 & J_1 - iJ_2 & 0 & 0 \\ J_1 + iJ_2 & -J_3 & 0 & 0 \end{pmatrix} \tag{25}$$

We then have that each one of the nonnull columns of ψ satisfies a (matrix) Dirac-like equation. In particular, taking into account that the Maxwell equations are invariant under the substitutions $\mathbf{B} \mapsto \mathbf{E}$, $\mathbf{E} \mapsto -\mathbf{B}$ and considering a medium with dielectric constant ε and magnetic permeability μ, and with the substitution $B \to H$ in (24), putting c for the light velocity, we get that the free electromagnetic field satisfies

$$\left\{\boldsymbol{\gamma} \cdot \nabla - \begin{pmatrix} \varepsilon \mathbb{1}_2 & 0 \\ 0 & -\mu \mathbb{1}_2 \end{pmatrix} \frac{1}{c}\frac{\partial}{\partial t}\right\}\psi = 0 \tag{26}$$

which is the equation originally obtained by Sallhöfer (1986) in an *ad hoc* way.

We are not going to discuss the physical interpretation proposed by Sallhöfer for equation (26). We simply say that the Maxwell and Dirac equations are indeed different when the comparison is done in the right way as above.

(*iii*) *The Maxwell Field as an Algebraic Majorana Spinor.* We already considered $\mathscr{C}\ell(\bar{\mathscr{L}})$ in Section 2, where we showed that the Dirac field in $Cl(\bar{\mathscr{L}})$ is represented by the elements of a nonminimal ideal section. Now, Majorana spinors are the elements of the minimal $\mathbb{R}_{3,1}\bar{E}$, where $\bar{E}=\frac{1}{4}(1-\bar{E}_1)(1-\bar{E}_0\bar{E}_3)$. Considering the isomorphism

$$\begin{aligned}\bar{\gamma}\colon\ \mathbb{R}_{3,1} &\to \mathscr{L}_{\mathbb{R}}(\bar{I})\\ p\to\bar{\gamma}(p)\colon\ &\bar{I}\to\bar{I}\\ &\bar{\psi}\to p\bar{\psi}\end{aligned}\tag{27}$$

gives, through a technique introduced in Figueiredo *et al.* (1990) the following representation for $\bar{\gamma}(E_\mu)=\bar{\gamma}^\mu$:

$$\begin{aligned}\bar{\gamma}^0&=\begin{bmatrix}-i\sigma_2 & 0\\ 0 & i\sigma_2\end{bmatrix}; &\quad \bar{\gamma}^1&=\begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}\\ \bar{\gamma}^2&=\begin{bmatrix}0 & i\sigma_2\\ -i\sigma_2 & 0\end{bmatrix}; &\quad \bar{\gamma}^3&=\begin{bmatrix}-\sigma_1 & 0\\ 0 & -\sigma_1\end{bmatrix}\end{aligned}\tag{28}$$

which is the set of Majorana matrices used by Srivastrava (1985) in his presentation of supersymmetry.

Let $\{\bar{e}_\mu, \mu=0,1,2,3\}$ be a global tetrad field with $\bar{e}_\mu\bar{e}_\nu+\bar{e}_\nu\bar{e}_\mu=-2\eta_{\mu\nu}$ and $\{\bar{\gamma}^\mu, \bar{\mu}=0,1,2,3\}$ the associated coframe field. Then $\bar{\mathbf{e}}=\frac{1}{4}(1-\bar{\gamma}^1)(1-\bar{\gamma}^0\bar{\gamma}^3)$ is a global idempotent field of minimum rank 4. The cross section $\bar{\psi}\bar{\mathbf{e}}=\psi$ is said to be an algebraic Majorana spinor field. The Maxwell equations (with the obvious change due to the metric of $\bar{\mathscr{L}}$) are now $\partial\bar{\psi}=\bar{\chi}$, $\bar{\psi}=F\bar{\mathbf{e}}$, $\chi=J\bar{\mathbf{e}}$. In the spinorial basis generated by (28) we have

$$\begin{aligned}\bar{\psi}&=\begin{pmatrix}E_3 & 0 & 0 & 0\\ E_1+B_2 & 0 & 0 & 0\\ E_2-B_1 & 0 & 0 & 0\\ B_3 & 0 & 0 & 0\end{pmatrix}\\ \bar{\chi}&=\begin{pmatrix}J_1 & 0 & 0 & 0\\ -J_0-J_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ J_2 & 0 & 0 & 0\end{pmatrix}\end{aligned}\tag{29}$$

4. TRANSFORMATION LAWS OF DIRAC AND MAXWELL FIELDS AS ALGEBRAIC SPINOR FIELDS

It is a well-known result that the minimal left ideals of $I_{pq}=\mathbb{R}_{pq}e_{pq}$ [where e_{pq} is a minimal idempotent of $\mathbb{R}_{p,q}$, the real Clifford algebra associated with the vector space $\mathbb{R}^{p,q}$ of signature (p, q)] are representation modules of $\mathbb{R}_{p,q}$.

We now discuss the problem of equivalence of these representations. To this end, remembering that $\mathbb{R}_{p,q}$ is not just an algebra, but an algebraic structure consisting of an algebra together with a distinguished subspace $\mathbb{R}^1_{p,q} \simeq \mathbb{R}^{p,q}$ and that the representation spaces I_{pq} are certain subalgebras of $\mathbb{R}_{p,q}$, we have the following theorems (Rodrigues *et al.*, 1989*a*; Bleecker, 1971; Porteous, 1981).

Theorem of Noether-Skolen. When $\mathbb{R}_{p,q}$ is simple, its automorphism is given its *inner* automorphism $m \mapsto umu^{-1}$, $m \in \mathbb{R}_{p,q}$, and $u = \Gamma(p, q)$, the Clifford group.[7]

Theorem. When $\mathbb{R}_{p,q}$ is simple, all its finite-dimensional irreducible representations are equivalent under inner automorphisms.

In view of the above theorems we define that two representations I_{pq} and I'_{pq} are equivalent if $I'_{pq} = uI_{pq}u^{-1}$ for some $u \in \Gamma(p, q)$.

Now taking into account the definition of the group $Spin_+(p, q)$, we see that the ideals I_{pq} can be made into spinorial representation of $SO_+(p, q)$ (*in the sense of group theory*) by postulating $I_{pq} \to uI_{pq}$ for $u \in Spin_+(p, q)$. This is exactly the idea behind the introduction of the spinorial metric introduced in Figueiredo *et al.* (1990).

The transformation $\psi \mapsto u\psi$, $\psi \in I_{pq}$ corresponds to the usual transformations of covariant spinors (Figueiredo *et al.*, 1990; Landau and Lifschitz, 1971), but the use of this transformation in a field formalism involving different Clifford fields beside algebraic spinor fields would contradict (if care is not taken) the fact that locally these spinors are elements of I_{pq}, which is a substructure of $\mathbb{R}_{p,q}$. This is why we need to consider two different bundles $\mathcal{C}\ell(\mathcal{L})$ and $S\mathcal{C}\ell(\mathcal{L})$. This will be discussed further below. Before we do that, let us observe that when $\mathbb{R}_{p,q}$ is semisimple and e_{pq} is a primitive idempotent, then $\mathbb{R}_{p,q}e_{pq}\mathbb{R}_{pq}$ is a bilateral ideal and since $1e_{pq}1 = e_{pq} \neq 0$ it follows that $\mathbb{R}_{p,q}e_{pq}\mathbb{R}_{p,q} = \mathbb{R}_{p,q}$. We can then write

$$\mathbb{R}_{p,q} = (I_{pq})(^*I_{pq}) \tag{30}$$

$$I_{pq} = \mathbb{R}_{p,q}e_{pq}, \qquad ^*I_{pq} = e_{pq}\mathbb{R}_{pq}$$

The meaning of (29) is then that $\forall X \in \mathbb{R}_{pq}$ can be written as sums of elements of the tensor product of the spinor spaces I_{pq} and $^*I_{pq}$, i.e., any $X \in \mathbb{R}_{p,q}$ can be considered as a rank-two spinor. This decomposition of antisymmetric tensors is the one generally presented in textbooks on theoretical physics and group theory and which gave birth to the belief that spinors are more fundamental than tensors (Frescura and Hiley, 1980;

[7] $\Gamma(p, q) = \{u | uu^{-1} = u^{-1}u = 1 \text{ and } Ad_u(\mathbb{R}^{p,q}) = R^{p,q},\ Ad_u x = uxu^{-1},\ x \in \mathbb{R}^{p,q}\}$

Penrose and Rindler, 1984). However, as is by now well known (Figueiredo *et al.*, 1990), $\forall X \in \mathbb{R}_{p,q}$ can be written as

$$X = \psi_1 + \psi_2 + \cdots + \psi_n \tag{31}$$

where $\psi_i \in I^i_{pq}$, $I^i_{pq} = e_{pq,i} \sum_{i=1}^n e_{pq,i} = 1$.

From equation (31) it is clear that algebraic spinors can be written as sums of antisymmetric tensors, as we know from the examples in Sections 2 and 3.

Now let $\{e_{pq,i};\ i = 1, \ldots, n\}$ and $\{e_{pq,i'};\ i' = 1, \ldots, n\}$ be two sets of primitive idempotents, $\sum_{i=1}^n e_{pq,i} = \sum_{i'=1}^n e_{pq,i'} = 1$ and $e_{pq,i'} = ue_{pq}u^{-1}$, $u \in \Gamma(p, q)$. Given $X \in \mathbb{R}_{p,q}$, we have

$$X = \sum_{i=1}^n Xe_{pq,i} = \sum_{i'=1}^n Xe_{pq,i'} = \psi_1 + \cdots + \psi_n = \psi' + \cdots + \psi'_n \tag{32}$$

$$\psi'_i = u\psi_i u^{-1} \tag{33}$$

On the other hand, if a given $X \in \mathbb{R}_{p,q}$ can be written as

$$X = \psi * \varphi; \qquad \psi \in I_{pq},\ {}^*\varphi \in {}^*I_{pq} \tag{34}$$

we can write

$$\begin{aligned} X = \psi^*\varphi &= \sum_\alpha \psi_\alpha e_{pq} e_{pq}\, {}^*\varphi_\alpha; \qquad \psi_\alpha \in I_{pq}, \quad {}^*\varphi_\alpha \in {}^*I_{pq} \\ &= \sum_\alpha \psi_\alpha (u^{-1} e_{pq'} u)(u^{-1} e'_{pq} u)^* \varphi\alpha \\ &= \sum_\alpha (u^{-1}\psi'_\alpha e'_{pq}) uu^{-1} (e'_{pq}\, {}^*\varphi'_\alpha u); \qquad \psi'_\alpha = u\psi_\alpha u^{-1}, \quad {}^*\varphi'_\alpha = u^*\varphi_\alpha u^{-1} \\ &= u^{-1} \sum_\alpha (\psi'_\alpha e'_{pq}) uu^{-1} (e'_{pq}\, {}^*\varphi'_\alpha) u \end{aligned} \tag{35}$$

In this case the factor uu^{-1} can be eliminated or retained without affecting the result for the decomposition of X into two distinct sums of products of spinors belonging to different ideals. Then, from the usual decomposition of an antisymmetric tensor as the tensor product of spinors, we see that the transformation law of spinors can be chosen either as

$$\psi \mapsto \psi' = u\psi \tag{36a}$$

or

$$\psi \mapsto \psi' = u\psi u^{-1} \tag{36b}$$

Equation (36b) results from the "sum decomposition" of X into spinors. Observe that if $\psi \in I_{pq}$, then $u\psi \in I_{pq}$, but $u\psi u^{-1} \notin I_{pq}$ in general.

In physical theories which use covariant spinor fields defined as sections of the covariant spinor bundle $\bar{S}(\mathscr{L})$ the observables are usually associated with bilinear functions of spinors (i.e., the observables are tensors) and the choice of transformation (36a) or (36b) is irrelevant. Whereas Aharonov and Susskind (1967) say that the transformation (34a) can be directly observed, their arguments are not very strong, mainly due to the prejudice that spinors are more fundamental than tensors. We come back to this point into another paper.

We can finally discuss the transformation laws of the Dirac and Maxwell fields considered as algebraic spinor fields, and the covariance of Dirac and Maxwell equations for these fields. As we saw in Sections 2 and 3, algebraic spinor fields can be thought of as sections $\psi e = \psi$ of $\mathscr{C}\ell(\mathscr{L})$ or $S\mathscr{C}\ell(\mathscr{L})$. Now, the structure of these bundles is as follows.

(A) The (real) spin-Clifford bundle is the bundle (Rodrigues and Figueiredo, 1990; Blaine Lawson and Michelsohn, 1983)

$$S\mathscr{C}\ell(\mathscr{L}) = P_{Spin_+(1,3)} \times_l \mathbb{R}_{1,3} \tag{37}$$

As $\mathbb{R}_{1,3}$ can be considered as a module over itself, the action of $Spin_+(1,3)$ is the usual left action, here done by left multiplication. $S\mathscr{C}\ell(\mathscr{L})$ is a "principal $\mathbb{R}_{1,3}$ bundle," i.e., it admits a free action of $\mathbb{R}^+_{1,3}$ on the right. There is a natural embedding $P_{Spin_+(1,3)}(\mathscr{L}) \subset S\mathscr{C}\ell(\mathscr{L})$ which comes from the embedding $Spin_+(p,q) = \{u \in \mathbb{R}^+_{p,q} \mid u^* u = 1\}$ for $p+q \leq 5$.

Hence, every real spinor bundle for $\mathscr{L}$ can be captured from this one.

(B) The Clifford bundle is the bundle (Rodrigues and Figueiredo, 1990; Graf, 1978; Blaine Lawson and Michelsohn, 1983)

$$\mathscr{C}\ell(\mathscr{L}) = P_{SO_+(1,3)} \times_{\rho_c} \mathbb{R}_{1,3} \tag{38}$$

where

$$\rho_c: \quad SO_+(1,3) \to Aut(\mathbb{R}_{1,3})$$

Now, if we remember that there exists the representation

$$Ad: \quad Spin_+(1,3) \to Aut(\mathbb{R}_{1,3})$$

given by $Ad_u X = uXu^{-1}$ for $u \in Spin_+(1,3)$ and $X \in \mathbb{R}_{1,3}$ so that $Ad_{-1} = Id = 1$, we see that this representation reduces to a representation of $SO_+(1,3)$ that is exactly ρ_c. Then, if the manifold $\mathscr{L}$ admits a spin structure (which is the case in the present paper), we can also write

$$\mathscr{C}\ell(\mathscr{L}) = P_{Spin_+(1,3)}(\mathscr{L}) \times_{ad} \mathbb{R}_{1,3} \tag{38'}$$

This shows the difference between $S\mathscr{C}\ell(\mathscr{L})$ and $\mathscr{C}\ell(\mathscr{L})$.

The result for the transformation law of the Dirac and Maxwell fields interpreted as algebraic spinors, i.e., as sections of $S\mathscr{C}\ell(\mathscr{L})$ or $\mathscr{C}\ell(\mathscr{L})$, is now clear.

When e is a global idempotent field of $S\mathcal{C}\ell(\mathcal{L})$ the transformation law of $\psi e = \psi$ is as follows.

Let $\phi_\alpha : \pi_{S\mathcal{C}\ell}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}_{1,3}$ and $\phi_\beta : \pi_{S\mathcal{C}\ell}^{-1}(U_\beta) \to U_\beta \times \mathbb{R}_{1,3}$ be two local trivializations of $S\mathcal{C}\ell(\mathcal{L})$. Let $\Psi: \mathcal{L} \supset U \to \pi_{S\mathcal{C}\ell}^{-1}(U)$ be a local section $\Psi = (x, \psi(x))$, $\psi(x)e = \psi(x)$.

The homeomorphisms ϕ_α and ϕ_β have the forms

$$\begin{gathered}\phi_\alpha(\Psi(x)) = (\pi_{S\mathcal{C}\ell}(\Psi(x)), \qquad \tilde{\phi}_\alpha(\psi(x))) \equiv (x, \psi_\alpha(x)) \\ \psi_\alpha(x) = \tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} \psi_\beta(x) = u\psi_\beta(x) \\ u = \tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} \in Spin_+(1,3); \qquad \psi_\alpha, \psi_\beta \in \mathbb{R}_{1,3}E\end{gathered} \tag{39}$$

where E is the idempotent in the typical fiber corresponding to e (see Sections 2 and 3).

The transformation law of an algebraic spinor field considered as an ideal section of $\mathcal{C}\ell(\mathcal{L})$ is as follows. Let $\phi_\alpha : \pi_c^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}_{1,3}$ and $\phi_\beta : \pi_c^{-1}(U_\beta) \to U_\beta \times \mathbb{R}_{1,3}$ be two local trivializations of $\mathcal{C}\ell(\mathcal{L})$. Let $\Psi: \mathcal{L} \supset U \to \pi_c^{-1}(U)$ be a local section with $\Psi = (x, \psi(x))$, $\psi(x)e = \psi(x)$.

The homeomorphisms ϕ_α and ϕ_β now have the forms

$$\begin{gathered}\phi_\alpha(\Psi(x)) = (\pi_c(\Psi(x)), \qquad \tilde{\phi}_\alpha(\psi(x))) = (x, \psi_\alpha(x)) \\ \psi_\alpha(x) = \tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} \psi_\beta(x) \equiv u\psi_\beta(x)u^{-1} \equiv Ad_u \psi_\beta(x)\end{gathered} \tag{40}$$

where $u \in Spin_+(1,3)$ and $\psi_\alpha, \psi_\beta \in \mathbb{R}_{1,3}E$.

Equations (39) and (40) show then that the transformation law of algebraic spinor fields depends explicitly on the *assumption* of whether they are sections of $\mathcal{C}\ell(\mathcal{L})$ or $S\mathcal{C}\ell(\mathcal{L})$. When $\mathcal{L}$ is a flat manifold, the Dirac and Maxwell equations in $\mathcal{C}\ell(\mathcal{L})$ or $S\mathcal{C}\ell(\mathcal{L})$ are equivalent. Note that term $qA\phi$ in (15) transforms under the homeomorphism $\phi_\alpha \circ \phi_\beta^{-1}$ (corresponding to a Lorentz transformation) as $qu(A\phi) = quAu^{-1}u\phi = qA'\phi'$. This makes the equation covariant and gives the right transformation law for A $(A \mapsto uAu^{-1})$ if either $\phi \in \sec \mathcal{C}\ell(\mathcal{L})$ or $\phi \in \sec S\mathcal{C}\ell(\mathcal{L})$.

5. CONCLUSIONS

In this paper we showed how the Dirac and Maxwell equations can be written in the Clifford $[\mathcal{C}\ell(\mathcal{L})]$ and spin-Clifford $[S\mathcal{C}\ell(\mathcal{L})]$ bundles over space-time.

Our approach makes possible a direct comparison of the Dirac and Maxwell equations. We also obtained all presentations of the Maxwell fields as covariant spinor fields appearing in the literature by appropriate choice of the global idempotent fields in $\mathcal{C}\ell(\mathcal{L})$ or $S\mathcal{C}\ell(\mathcal{L})$. The transformation laws for the Dirac and Maxwell fields represented as algebraic spinors fields is clarified. It is important to emphasize that when $\mathcal{L}$ is a flat manifold, so that $S\mathscr{C}\ell(\mathscr{L})$ exists, there is no way to decide on the use of $\mathscr{C}\ell(\mathscr{L})$ or $S\mathscr{C}\ell(\mathscr{L})$ for the description of the Dirac and Maxwell fields. For a general Lorentzian manifold, $\mathscr{C}\ell(\mathscr{L})$ always exists, but the obvious generalization of the Dirac equation for an algebraic spinor field that is an ideal section of $\mathscr{C}\ell(\mathscr{L})$ implies some obstructions for the manifold $\mathscr{L}$ (Graf, 1978). Also, for a general Lorentzian manifold the Dirac equation will be different in $\mathscr{C}\ell(\mathscr{L})$ and $S\mathscr{C}\ell(\mathscr{L})$ due to the existence of the spin connection in $S\mathscr{C}\ell(\mathscr{L})$ (Blaine Lawson and Michelsohn, 1983). Thus, any solution to the real nature of Dirac's algebraic spinor field as a section of $\mathscr{C}\ell(\mathscr{L})$ or $S\mathscr{C}\ell(\mathscr{L})$ must be done in strong gravitational fields. This is a delicate problem, to which we will return in another paper.

To end this paper, we observe that Oppenheimer (1931) and also Majorana (see Mignani *et al.*, 1974) wrote the Maxwell equations using 3×3 spinlike matrices, then putting the Maxwell equations in a Schrödinger-like form. The relation between the 3×3 matrices used by Oppenheimer and Majorana and the Dirac matrices has been discussed by Gianneto (1985).

ACKNOWLEDGMENTS

The authors are grateful to Dr. M. A. Faria-Rosa for useful discussions and to Prof. E. Recami for several comments and for pointing out several important references related to the issues addressed in the present paper. We are grateful to CNPq and FAPESP for research grants.

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